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A Relation Between a.c. Spectrum of Ergodic Jacobi Matrices and the Spectra of Periodic Approximants*

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Abstract. We study ergodic Jacobi matrices on $l^2(Z)$, and prove a general theorem relating their a.c. spectrum to the spectra of periodic Jacobi matrices, that are obtained by cutting finite pieces from the ergodic potential and then repeating them. We apply this theorem to the almost Mathieu operator: $(H_{\alpha,\lambda,\theta}u)(n) = u(n+1) + u(n-1) + \lambda\cos(2\pi\alpha n + \theta)u(n)$, and prove the existence of a.c. spectrum for sufficiently small λ, all irrational α's, and a.e. θ. Moreover, for $0 \le \lambda < 2$ and (Lebesgue) a.e. pair α, θ, we prove the explicit equality of measures: $|\sigma_{\text{ac}}| = |\sigma| = 4 - 2\lambda$.

1. Introduction

In this paper, we study one dimensional ergodic Jacobi matrices. These are families of (bounded, self adjoint) operators H_ω on $l^2(Z)$, defined by:

$$H_\omega = H_0 + V_\omega\,, \qquad (H_0u)(n) = u(n+1) + u(n-1)\,, \qquad (V_\omega u)(n) = V_\omega(n)u(n)\,, \tag{1.1}$$

where V_ω is a (real) stationary bounded ergodic potential, that is: we consider a probability measure space (Ω, dp), a measure preserving invertible ergodic transformation T, and a bounded measurable real-valued function f, and define: $V_\omega(n) = f(T^n\omega)$.

For such a family $\{H_\omega\}_{\omega\in\Omega}$, it is known [9] that the spectrum of H_ω, and its decomposition into a.c., s.c., and p.p. parts are a.e. constant with respect to ω. Namely, there are subsets: $\sigma, \sigma_{\text{ac}}, \sigma_{\text{sc}}, \sigma_{\text{pp}}$ of R, such that for a.e. $\omega : \sigma_\omega \equiv \text{Spec}(H_\omega) = \sigma$ and $\sigma_{\text{ac}}, \sigma_{\text{sc}}, \sigma_{\text{pp}}$ are (respectively) the absolutely continuous, singular continuous and pure point spectra of H_ω (σ_{pp} being the closure of the set of eigenvalues).

The Lyapunov exponent $\gamma(E)$, characterizing solutions of the equation:

$$u(n+1) + u(n-1) + V_\omega(n)u(n) = Eu(n) \tag{1.2}$$

is defined for the family $\{H_\omega\}_{\omega\in\Omega}$, as follows:

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