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The Newtonian Limit of the Spherically Symmetric Vlasov–Einstein System

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Abstract. We prove that spherically symmetric solutions of the Vlasov–Einstein system with a fixed initial value converge to the corresponding solution of the Vlasov–Poisson system if the speed of light c is taken as a parameter and tends to infinity. The convergence is uniform on compact time intervals with convergence rate $1/c^2$. Thus the classical Vlasov–Poisson system appears as the Newtonian limit of the general relativistic Vlasov–Einstein system in a spherically symmetric setting.

1. Introduction

Consider an ensemble of particles (stars in a galaxy, galaxies in a galaxy cluster, etc.) which are all of the same mass and interact only by the gravitational field which they generate collectively. The ensemble is described by a time dependent density function f on phase space, and this function satisfies a continuity equation, the Vlasov or Liouville equation, which is coupled to the field equations with source terms generated by f. In the classical, Newtonian setting we obtain the Vlasov–Poisson system (VP)

$$\partial_t f + v \cdot \partial_x f - K(t,x) \cdot \partial_v f = 0 ,$$

$$K(t,x) := \frac{x}{r}\frac{1}{r^2} \int_{|y| \leqq r} \int_{\mathbb{R}^3} f(t,y,v)\, dv\, dy ,$$

where $t \geqq 0$ denotes time, $x \in \mathbb{R}^3$ position, and $v \in \mathbb{R}^3$ momentum, $r := |x|$, and we have assumed that the system is spherically symmetric, i.e. $f(t, Ax, Av) = f(t, x, v)$, $K(t, Ax) = AK(t, x)$ for any orthogonal matrix A and $t \geqq 0, x, v \in \mathbb{R}^3$.

If we wish to describe the above situation in the setting of general relativity, we obtain the Vlasov–Einstein system (VEγ) in the following form:

$$\partial_t f + e^{\mu-\lambda} \frac{v}{\sqrt{1+\gamma v^2}} \cdot \partial_x f - \left(\dot{\lambda} \frac{x \cdot v}{r} + e^{\mu-\lambda} \frac{1}{\gamma} \mu' \sqrt{1+\gamma v^2} \right) \frac{x}{r} \cdot \partial_v f = 0 ,$$

$$e^{-2\lambda}(2r\lambda' - 1) + 1 = 8\pi\gamma r^2 \rho ,$$

$$e^{-2\lambda}(2r\mu' + 1) - 1 = 8\pi\gamma^2 r^2 p ,$$