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Chern–Simons Solitons, Toda Theories and the Chiral Model*

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Abstract. The two-dimensional self-dual Chern–Simons equations are equivalent to the conditions for static, zero-energy solutions of the $(2 + 1)$-dimensional gauged nonlinear Schrödinger equation with Chern–Simons matter-gauge dynamics. In this paper we classify all finite charge $SU(N)$ solutions by first transforming the self-dual Chern–Simons equations into the two-dimensional chiral model (or harmonic map) equations, and then using the Uhlenbeck–Wood classification of harmonic maps into the unitary groups. This construction also leads to a new relationship between the $SU(N)$ Toda and $SU(N)$ chiral model solutions.

1. Introduction

The study of the nonlinear Schrödinger equation in $2 + 1$-dimensional space-time is partly motivated by the well-known successes of the $1 + 1$-dimensional nonlinear Schrödinger equation. Here we consider a *gauged* nonlinear Schrödinger equation in which we have not only the nonlinear potential term for the matter fields, but also we have a coupling of the matter fields to gauge fields. Furthermore, this matter-gauge dynamics is chosen to be of the Chern–Simons form rather than the conventional Yang–Mills form. Such a choice is motivated by the fact that the resulting Schrödinger equation is related to a non-relativistic field theory for the many-body anyon system.

The theory with an Abelian gauge field was analyzed by Jackiw and Pi [11] who found static, zero energy solutions which arise from a two-dimensional notion of self-duality. The static, self-dual matter density satisfies the Liouville equation which is known to be integrable and, indeed, solvable in the sense that the general (real) solution may be expressed in terms of an arbitrary holomorphic function [14]. The gauged nonlinear Schrödinger equation with a *non-Abelian* Chern–Simons matter-gauge dynamics has also been considered [8, 4, 5], and once

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