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Rigorous Results for the Free Energy in the Hopfield Model

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Abstract. We prove that the free energy of the Hopfield model with a finite number of patterns can be represented in terms of an asymptotic series expansion in inverse powers of the neurons number. The series is Borel summable for large temperatures. We also establish mathematically some other interesting properties, partly used before in a seminal paper by Amit, Gutfreund and Sompolinsky.

1. Introduction

One of the first papers in which the critical temperature of the Hopfield model and the properties of the overlaps were intensively discussed is "Spin-glass models of neural networks" by D.J. Amit, H. Gutfreund and H. Sompolinsky [1]. In the first part of that paper the expression of the free energy was deduced in the limit $N \to \infty$ applying heuristically the saddle point technique to the expression of the partition function, for $N < \infty$. The results of that calculation were very interesting and many of them have been checked also by numerical simulations. Obviously it was not the aim of that paper to prove mathematically the results presented. In the present paper, besides providing mathematical proofs, we also deduce new stronger properties of the free energies. Let us briefly describe our main results, referring to Sects. 2 and 3 for proofs. Consider the Hamiltonian of the Hopfield model [8] with p patterns:

$$H_N(\xi) = \frac{-1}{2N} \sum_{\mu=1}^{p} \sum_{i \neq j, i, j=1}^{N} \xi_i^\mu \xi_j^\mu \sigma_i \sigma_j , \tag{1.1}$$

where σ_i, $i = 1, \ldots, N$ are the neuronal activities, $\sigma_i = \pm 1$, and

$$\xi_i^\mu, \quad i = 1, \ldots, N, \quad \xi_i^\mu = \pm 1$$

is the codification of the μ^{th} pattern which we want to memorize, $\mu = 1, \ldots, p$.

All the ξ_i^μ are considered to be random variables independent and equally distributed with probability $E\{\xi_i^\mu = \pm 1\} = \frac{1}{2}$. The retrieval property of the Hopfield model corresponds to the fact that the "mean value" of the "spins" is