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Towards a Rigorous Treatment of the Jeans–Landau–Teller Method for the Energy Exchanges of Harmonic Oscillators

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Abstract. For classical Hamiltonian systems containing an harmonic oscillator of high frequency, one has the problem of controlling the energy exchange between the oscillator and the remaining "slow" degrees of freedom; under very general conditions, such an exchange turns out to be exponentially small with the frequency of the oscillator. In the Jeans–Landau–Teller method, one aims to prove the exponential dependence, and to estimate the coefficient of the exponential, by exploiting the analyticity properties of the solution of the differential equations describing the motion of the system. However, in practice, since the exact solution is not known, such properties are inferred from those of an approximate solution, with no control of the difference; this fact might a priori even invalidate the exponential dependence itself. In the present paper a rigorous treatment is given, for a particular model of interest in the domain of atomic collisions, by keeping control of the difference between the exact and the approximate solution.

1. Introduction

The problem of estimating the exchanges of energy for systems containing harmonic oscillators of high frequency is a very relevant one in many domains of physics. A rigorous mathematical treatment was given, rather recently, by Neishtadt [1] and by Benettin, Galgani and Giorgilli [2, 3], by adapting the methods of classical perturbation theory developed by Nekhoroshev [4, 5]. A typical result is that the exchange of energy between a system of harmonic oscillators of the same angular frequency ω and another system is, under quite general conditions, exponentially small with ω:

$$\Delta E \leqq C e^{-a\omega}, \quad C, a \geqq 0\,, \tag{1.1}$$

if ω is sufficiently large. Unfortunately, as is typical of perturbation theory, the estimates one gets for the constants, in particular for a, which is the most relevant one, are terribly pessimistic and, as could be seen by numerical computations [6] (see also [7] and the papers there quoted), very far from being realistic. Moreover,