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On Geometric Phases for Soliton Equations

M.S. Alber[1] and J.E. Marsden[2]

[1] Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA
[2] Department of Mathematics, University of California, Berkeley, CA 94720, USA

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Abstract. This paper develops a new complex Hamiltonian structure for n-soliton solutions for a class of integrable equations such as the nonlinear Schrödinger, sine-Gordon and Korteweg-de Vries hierarchies of equations that yields, amongst other things, geometric phases in the sense of Hannay and Berry. For example, one of the possible soliton geometric phases is manifested by the well known phase shift that occurs for interacting solitons. The main new tools are complex angle representations that linearize the corresponding Hamiltonian flows on associated non-compact Jacobi varieties. This new structure is obtained by taking appropriate limits of the differential equations describing the class of quasi-periodic solutions. A method of asymptotic reduction of the angle representations is introduced for investigating soliton geometric phases that are related to the presence of monodromy at singularities in the space of parameters. In particular, the phase shift of interacting solitons can be expressed as an integral over a cycle on an associated Riemann surface. In this setting, soliton geometric asymptotics are constructed for studying geometric phases in the quantum case. The general approach is worked out in detail for the three specific hierarchies of equations mentioned. Some links with τ-functions, the braid group and geometric quantization are pointed out as well.

1. Introduction

We begin by summarizing a few recent developments in the theory of geometric phases that are relevant to the present paper. In [12] Berry considered a geometric phase factor $\exp(i\gamma)$ (in addition to the dynamical phase) for systems that are slowly transported along a closed curve in a space of parameters. In [41] a class of connections was constructed to obtain expressions for the Hannay–Berry phases [14] (geometric angle shifts in the classical case) for some integrable problems in terms of the non-trivial holonomy of these connections. In [41], Montgomery gave an example of a phase that is linked with singularities in the case of a flat connection. Symmetry and reduction were used to obtain a generalization of