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# The Logarithmic Sobolev Inequality for Discrete Spin Systems on a Lattice*

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**Abstract.** For finite range lattice gases with a finite spin space, it is shown that the Dobrushin–Shlosman mixing condition is equivalent to the existence of a logarithmic Sobolev inequality for the associated (unique) Gibbs state. In addition, implications of these considerations for the ergodic properties of the corresponding Glauber dynamics are examined.

## 1. Preliminaries

We begin by introducing the setting in which and some of the notation with which we will be working throughout.

**The Lattice.** The lattice $\Gamma$ underlying our model will be the $d$-dimensional square lattice $\mathbb{Z}^d$ for some fixed $d \in \mathbb{Z}^+$, and, for $\mathbf{j} \in \Gamma$, we will use the norm $|\mathbf{k}| \equiv \max_{1 \leqq i \leqq d} |\mathbf{k}^i|$. Given $\Lambda \subseteq \Gamma$, we will use $\Lambda\complement \equiv \Gamma \backslash \Lambda$ to denote the complement of $\Lambda$, $|\Lambda|$ to denote the cardinality of $\Lambda$, and $\mathbf{j} + \Lambda$ to denote the translate $\{\mathbf{j} + \mathbf{k} : \mathbf{k} \in \Lambda\}$ of $\Lambda$ by $\mathbf{j} \in \Gamma$. Furthermore, for each $R \in \mathbb{R}^+$, we take the *R-boundary* $\partial_R \Lambda$ to be the set

$$\{\mathbf{k} \in \Lambda\complement : |\mathbf{k} - \mathbf{j}| \leqq R \text{ for some } \mathbf{j} \in \Lambda\} .$$

We will often use the notation $\Lambda \Subset \Gamma$ to mean that $|\Lambda| < \infty$, and $\mathfrak{F}$ will stand for the set of all non-empty $\Lambda \Subset \Gamma$. A monotone sequence $\mathfrak{F}_0 \equiv \{\Lambda_n : n \in \mathbb{N}\} \subseteq \mathfrak{F}$ will be called *a countable exhaustion* if $\Lambda_n \nearrow \Gamma$.

**The Spin Space.** The single spin space for our model will be a finite set $Q$ with the topology of all subsets, corresponding Borel field $\mathscr{B}_Q$, and normalized uniform measure $\nu_0$ on $(Q, \mathscr{B}_Q)$. Given a real-valued function $f$ on $Q$, we define the *differential* $\partial f$ of $f$ by

$$\partial f \equiv f - \nu_0 f ,$$

where we have introduced the notation $\mu\varphi$ (to be used throughout) as one of the various expressions for the integral of a $\mu$-integrable function $\varphi$ with respect to a measure $\mu$.

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