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A Lie Theoretic Galois Theory for the Spectral Curves of an Integrable System: I

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Abstract. In the study of integrable systems of ODE's arising from a Lax pair with a parameter, the constants of the motion occur as spectral curves. The specific curves depend upon the representation of the Lie algebra. In this paper a Galois theory of spectral curves is given that classifies the spectral curves from an integrable system. The spectral curves correspond to conjugacy classes of certain subgroups of the Weyl group for the Lie algebra. The theory is illustrated with the periodic Toda lattice.

Introduction

One mechanism for producing constants of the motion for completely integrable systems is the Lax pairing. The idea introduced by Peter Lax was to express systems of differential equations in the form $\frac{dA}{dt} = [A, B]$. For finite dimensional systems the following desirable situation often occurs:

1. A and B lie in a Lie algebra $\mathfrak{g}$,
2. A and B are functions of time t and rational functions of a parameter s where s is a coordinate on an algebraic curve P.

For each representation ρ of $\mathfrak{g}$, the characteristic polynomial of $\rho(A)$ defines a curve by the equation $0 = \det(\rho A(s, t) - z)$. It is a consequence of the Lax form of the differential equation for A that $\{(s, z) \mid 0 = \det(\rho A(s, t) - z)\}$ is independent of time for any representation ρ. The curves defined by $0 = \det(\rho A(s, t) - z)$ are in general reducible and the irreducible components of these various curves are called spectral curves. These curves are equipped with projections to P via the s coordinate. This arrangement has been used by many mathematicians to examine completely integrable systems, e.g., van Moerbeke and Mumford [vMM]; Adler and van Moerbeke [AvM1, 2]; McDaniel [Mc]; Kanev [K]; Griffiths [G];

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