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A Stochastic Theory of Adiabatic Invariance

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Abstract. Let I be a set of invariants for a system of differential equations with an order $o(\varepsilon)$ vector field. When order ε perturbations of zero mean are added to the system we show that, under suitable regularity and ergodicity conditions, I becomes an adiabatic invariant with maximal variations of order one on time scales of order $1/\varepsilon^2$. In the stochastically perturbed case, I behaves asymptotically (for small ε) like a diffusion process on $1/\varepsilon^2$ time scales. The results also apply to an interesting class of deterministic perturbations. This study extends the results of Khas'minskii on stochastically averaged systems, as well as some of the deterministic methods of averaging, to such invariants.

0. Introduction

We consider the behavior of the stochastic differential equation in R^p,

$$\dot{x} = \varepsilon F(x, t, \varepsilon, \omega), \quad x(0) = x_0 , \tag{0.1}$$

where

$$F(x, t, \varepsilon, \omega) = f(x, t) + F^{(0)}(x, t, \omega) + \varepsilon F^{(1)}(x, t, \omega) + o(\varepsilon) \tag{0.2}$$

as $\varepsilon \to 0$. We require that $EF^{(0)}(x, t) = 0$ and that

$$\bar{f}(x) = \lim_{t \to \infty} \frac{1}{t} \int_0^t f(x, s) ds \tag{0.3}$$

exists for all x. Khas'minskii [13] examined these systems at time scales of $0(1/\varepsilon)$ and gave general conditions for the asymptotic approximation of solutions of (0.1) by solutions of the deterministic equations

$$\dot{x} = \varepsilon \bar{f}(x) \tag{0.4}$$

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