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Realizability of a Model in Infinite Statistics

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Abstract. Following Greenberg and others, we study a space with a collection of operators $a(k)$ satisfying the "q-mutator relations" $a(l)a^\dagger(k) - qa^\dagger(k)a(l) = \delta_{k,l}$ (corresponding for $q = \pm 1$ to classical Bose and Fermi statistics). We show that the $n! \times n!$ matrix $A_n(q)$ representing the scalar products of n-particle states is positive definite for all n if q lies between -1 and $+1$, so that the commutator relations have a Hilbert space representation in this case (this has also been proved by Fivel and by Bożejko and Speicher). We also give an explicit factorization of $A_n(q)$ as a product of matrices of the form $(1 - q^jT)^{\pm 1}$ with $1 \leqq j \leqq n$ and T a permutation matrix. In particular, $A_n(q)$ is singular if and only if $q^M = 1$ for some integer M of the form $k^2 - k$, $2 \leqq k \leqq n$.

1. Introduction

In this paper we study the following object: a Hilbert space $\mathbf{H}$ together with a non-zero distinguished vector $|0\rangle$ (vacuum state) and a collection of operators a_k: $\mathbf{H} \to \mathbf{H}$ satisfying the commutation relations ("q-mutator relations")

$$a(l)\,a^\dagger(k) - q\,a^\dagger(k)\,a(l) = \delta_{k,l} \quad (\forall k,\, l) \tag{1}$$

and the relations

$$a(k)|0\rangle = 0 \quad (\forall k)\,. \tag{2}$$

Here q is a fixed real number and $a^\dagger(l)$ denotes the adjoint of $a(l)$. The statistics based on the commutation relation (1) generalizes classical Bose and Fermi statistics, corresponding to $q = 1$ and $q = -1$, respectively, as well as the intermediate case $q = 0$ suggested by Hegstrom and investigated by Greenberg [1]. The study of the general case was inititated by Polyakov and Biedenharn [2].

Our first main result is a realizability theorem saying that the object just described exists if $-1 < q < 1$. In view of (2), we can think of the $a(k)$ as annihilation operators and the $a^\dagger(k)$ as creation operators. As well as the 0-particle state $|0\rangle$, our space must contain the many-particle states obtained by applying combinations of $a(k)$'s and $a^\dagger(k)$'s to $|0\rangle$. To prove the realizability of our model it is obviously necessary and sufficient to consider the minimal space