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A Second Proof of the Payne–Pólya–Weinberger Conjecture

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Abstract. Without using product representations or elaborate comparisons of zeros we prove the two key properties of the Bessel function ratio $J_{p+1}(j_{p+1,1}x)/J_p(j_{p,1}x)$ that we used to prove the Payne–Pólya–Weinberger conjecture. In these new proofs we use only differential equations and the Rayleigh–Ritz method for estimating lowest eigenvalues. The new proofs admit generalization to other related problems where our previous proofs fail.

1. Introduction

In our recent proof [3, 4] of the Payne–Pólya–Weinberger conjecture [12, 13], the success of our method turned on certain technical results concerning Bessel functions. To be specific, two key properties of the function

$$w(x) = \frac{J_{p+1}(\beta x)}{J_p(\alpha x)} \quad \text{for} \quad x \in [0, 1] \tag{1.1}$$

are used (at $x = 0$ and $x = 1$ we define $w(x)$ by its limiting values from within $(0, 1)$), where $\alpha = j_{p,1}$ and $\beta = j_{p+1,1}$ denote the first positive zeros of the Bessel functions $J_p(x)$ and $J_{p+1}(x)$ (our notation follows that of Abramowitz and Stegun [1]). The two properties are

(1) $w(x)$ is increasing for $x \in [0, 1]$, and
(2) $B(x) \equiv w'(x)^2 + (2p + 1)w(x)^2/x^2$ is decreasing for $x \in [0, 1]$.

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