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Rapidly Decaying Solutions of the Nonlinear Schrödinger Equation

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Abstract. We consider global solutions of the nonlinear Schrödinger equation

$$iu_t + \Delta u = \lambda |u|^\alpha u, \quad \text{in } \mathbf{R}^N, \tag{NLS}$$

where $\lambda \in \mathbf{R}$ and $0 < \alpha < \frac{4}{N-2}$. In particular, for $\alpha > \alpha_0 = \frac{2 - N + \sqrt{N^2 + 12N + 4}}{2N}$, we show that for every $\varphi \in H^1(\mathbf{R}^N)$ such that $x\varphi(x) \in L^2(\mathbf{R}^N)$, the solution of (NLS) with initial value $\varphi(x)e^{i(b|x|^2/4)}$ is global and rapidly decaying as $t \to \infty$ if b is large enough. Furthermore, by applying the pseudo-conformal transformation and studying the resulting nonautonomous nonlinear Schrödinger equation, we obtain both new results and simpler proofs of some known results concerning the scattering theory. In particular, we construct the wave operators for $\frac{4}{N+2} < \alpha < \frac{4}{N-2}$. Also, we establish a low energy scattering theory for the same range of α and show that, at least for $\lambda < 0$, the lower bound on α is optimal. Finally, if $\lambda > 0$, we prove asymptotic completeness for $\alpha_0 \leqq \alpha < \frac{4}{N-2}$.

1. Introduction

In this paper we study solutions in $\mathbf{R}^N$ of the nonlinear Schrödinger equation

$$iu_t + \Delta u = \lambda |u|^\alpha u. \tag{1.1}$$

Here $u = u(t, x)$ is a complex valued function defined for t in some subset of the real numbers and all $x \in \mathbf{R}^N$, $\lambda \in \mathbf{R}$, and $0 < \alpha < \frac{4}{N-2}$. We frequently write $u(t)$ for the