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Completely Integrable Gradient Flows

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Abstract. In this paper we exhibit the Toda lattice equations in a double bracket form which shows they are gradient flow equations (on their isospectral set) on an adjoint orbit of a compact Lie group. Representations for the flows are given and a convexity result associated with a momentum map is proved. Some general properties of the double bracket equations are demonstrated, including a discussion of their invariant subspaces, and their function as a Lie algebraic sorter.

0. Introduction

In this paper we present details of the proofs and applications of the results on the generalized Toda lattice equations announced in [7].

One of our key results is exhibiting the Toda equations in a double bracket form which shows directly that they are gradient flow equations (on their isospectral set) on an adjoint orbit of a compact Lie group. This system, which is gradient with respect to the normal metric on the orbit, is quite different from the representation of the Toda flow as a gradient system on $\mathbb{R}^n$ in Moser's fundamental paper [27]. In our representation the same set of equations is thus Hamiltonian and a gradient flow on the isospectral set.

While the double bracket equations mentioned above are fundamental in demonstrating properties of the Toda flow, they are in fact more general and are of interest in their own right. These equations arose originally in the study by Brockett (see [10 and 11]) of the steepest descent equations corresponding to certain least squares matching and sorting problems. In [5] it was noticed that a

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