Commun. Math. Phys. 147, 25–56 (1992)

Communications in
Mathematical
Physics

A Matrix Integral Solution to two-dimensional W_p-Gravity

M. Adler [1] **and P. van Moerbeke** [1,2]

[1] Department of Mathematics, Brandeis University, Waltham, Mass 02254, USA
[2] Department of Mathematics, University of Louvain, B-1348-Louvain-la-Neuve, Belgium

Received October 22, 1991; in revised form January 17, 1992

Abstract. The p^{th} Gel'fand-Dickey equation and the string equation $[L, P] = 1$ have a common solution τ expressible in terms of an integral over $n \times n$ Hermitean matrices (for large n), the integrand being a perturbation of a Gaussian, generalizing Kontsevich's integral beyond the KdV-case; it is equivalent to showing that τ is a vacuum vector for a $\mathcal{W}_p^+$-algebra, generated from the coefficients of the vertex operator. This connection is established via a quadratic identity involving the wave function and the vertex operator, which is a disguised differential version of the Fay identity. The latter is also the key to the spectral theory for the two compatible symplectic structures of KdV in terms of the stress-energy tensor associated with the Virasoro algebra.

Given a differential operator

$$L = D^p + q_2(t)\, D^{p-2} + \cdots + q_p(t), \quad \text{with} \quad D = \frac{\partial}{dx}, \; t = (t_1, t_2, t_3, \ldots), \; x \equiv t_1,$$

consider the deformation equations [1]

$$\frac{\partial L}{\partial t_n} = [(L^{n/p})_+, L] \qquad n = 1, 2, \ldots, n \neq 0 \,(\operatorname{mod} p) \tag{0.1}$$

(p-reduced KP-equation)

of L, for which there exists a differential operator P (possibly of infinite order) such that

$$[L, P] = 1 \quad \text{(string equation)}. \tag{0.2}$$

In this note, we give a complete solution to this problem. In section 1 we give a brief survey of useful facts about the I-function $\tau(t)$, the wave function $\Psi(t, z)$, solution of $\partial \Psi / \partial t_n = (L^{n/p})_x \Psi$ and $L^{1/p} \Psi = z \Psi$, and the corresponding infinite-dimensional plane V^0 of formal power series in z (for large z)

$$V^0 = \operatorname{span}\{\Psi(t, z) \text{ for all } t \in \mathbb{C}^\infty\}$$

[1] $\left(\sum_{-\infty}^{\infty} b_i D_i\right)_+ = \sum_0^{\infty} b_i D_i, \; (\sum b_i D^i)_- = \sum_{-\infty}^{-1} b_i D_i, \; (\sum b_i D^i)_j = b_j.$