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Floquet Solutions for the 1-Dimensional Quasi-Periodic Schrödinger Equation

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Abstract. We show that the 1-dimensional Schrödinger equation with a quasi-periodic potential which is analytic on its hull admits a Floquet representation for almost every energy E in the upper part of the spectrum. We prove that the upper part of the spectrum is purely absolutely continuous and that, for a generic potential, it is a Cantor set. We also show that for a small potential these results extend to the whole spectrum.

1. Introduction

In this paper we will consider the Schrödinger equation

$$(*) \qquad (\mathscr{L}y)(t) = -y''(t) + q(\omega t) = Ey(t)$$

for a real *quasi-periodic* potential $q(\omega t)$ with frequency vector ω, and for large energies E or small potential q. We will study the existence and non-existence of *Floquet solutions* or *Bloch waves*, i.e. solutions of the form $y(t) = e^{kt}(p_1(t) + tp_2(t))$, where k is a constant and p_1, p_2 are quasi-periodic functions with the frequency vector ω or $\frac{\omega}{2}$. We will also study the nature of the spectrum $\sigma(\bar{\mathscr{L}})$, where $\bar{\mathscr{L}}$ is the closure of the operator

$$\mathscr{L}: C_c''(\mathbf{R}) \to L^2(\mathbf{R})$$

in the space $L^2(\mathbf{R})$ of complex square integrable functions on $\mathbf{R}$.

We shall assume that $q: \mathbf{T}^d \to \mathbf{R}$, $\mathbf{T} = \mathbf{R}/(2\pi\mathbf{Z})$, is analytic in a complex neighbourhood $|\operatorname{Im} x| < r$ of $\mathbf{T}^d$, and we shall use the norm

$$|q|_r = \sup_{|\operatorname{Im} x| < r} |q(x)|.$$

We shall also assume that ω is *diophantine*, i.e.

$$|\langle n \rangle| \geqq |n|^{-\tau}, \quad n \in \mathbf{Z}^d \setminus 0$$