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Cyclic Homology of Differential Operators, the Virasoro Algebra and a q-Analogue

Christian Kassel

Institut de Recherche Mathématique Avancée, Université Louis Pasteur – C.N.R.S.,
7 rue René Descartes, F-67084 Strasbourg, France

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Abstract. We show how methods from cyclic homology give easily an explicit 2-cocycle φ on the Lie algebra of differential operators of the circle such that φ restricts to the cocycle defining the Virasoro algebra. The same methods yield also a q-analogue of φ as well as an infinite family of linearly independent cocycles arising when the complex parameter q is a root of unity. We use an algebra of q-difference operators and q-analogues of Koszul and de Rham complexes to construct these "quantum" cocycles.

The Virasoro algebra Vir is the universal central extension of the Lie algebra $\mathrm{Der}(\mathbf{C}[x,x^{-1}])$ of derivations of the algebra $\mathbf{C}[x,x^{-1}]$ of complex Laurent polynomials. This extension

$$0 \to \mathbf{C} \to \mathrm{Vir} \to \mathrm{Der}(\mathbf{C}[x,x^{-1}]) \to 0$$

has a one-dimensional centre and is defined by the following 2-cocycle α on $\mathrm{Der}(\mathbf{C}[x,x^{-1}])$:

$$\alpha\left(P\frac{d}{dx}, Q\frac{d}{dx}\right) = \frac{1}{12}\mathrm{res}\begin{vmatrix} P' & Q' \\ P'' & Q'' \end{vmatrix} = \frac{1}{6}\mathrm{res}(QP''')$$

with $P, Q \in \mathbf{C}[x,x^{-1}]$. Here P' denotes the derived polynomial of P and res is the residue map. Set $L_n = x^{n+1}d/dx$; then the cocycle α takes the familiar form

$$\alpha(L_m, L_n) = \frac{m^3 - m}{6}\delta_{m+n,0},$$

where $\delta_{i,j}$ is the Kronecker symbol.

We now embed $\mathrm{Der}(\mathbf{C}[x,x^{-1}])$ in the associative algebra $\mathscr{D} = \mathrm{Diff}(\mathbf{C}[x,x^{-1}])$ of all algebraic differential operators on $\mathbf{C}[x,x^{-1}]$. The set $\{x^i(d/dx)^j\}_{i\in\mathbf{Z},\, j\in\mathbf{N}}$ is a basis of the complex vector space $\mathscr{D}$.

In [5] Kac and Peterson proved that the Virasoro algebra is a Lie subalgebra of a central extension of $\mathscr{D}$ considered as a Lie algebra (see also [8] for a generalization and [4] for related results). More precisely,