Commun. Math. Phys. 146, 73–90 (1992)

Communications in
Mathematical
Physics

On the Localization of Topological Invariants

I. M. Gelfand[1] **and B. L. Tsygan**[2]

[1] Moscow State University, Moscow, USSR and Rutgers University, New Brunswick, NJ 08903, USA
[2] Penn State University, University Park, PA 16802, USA

Received September 10, 1990; in revised form March 14, 1991

Abstract. The approach of formal differential geometry to the topological invariants which can be localized is developed. The universal space and universal characteristic forms are constructed. They give rise to primary and secondary characteristic forms.

Introduction

This paper is a continuation of the works [9–13] and develops the approach of the formal differential geometry. This approach asserts the following. First of all, many of the important topological invariants of a manifold may be localized, i.e., they may be obtained in the following way. Take a finite set of fields $\Gamma_1, \ldots, \Gamma_n$ on a smooth manifold; by a field we mean a differential geometric object having the prescribed transformation law under the action of the certain transformation group (for example, Riemannian metrics, connections in the given vector bundle, automorphisms of the bundle, various tensors etc.). Construct from the components of these fields and their derivatives some differential form in such a way that it should be covariant under the group of coordinate changes. The space of these forms with the covariance condition has the natural grading and differential. Indeed, the de Rham differential of such a form is again a form constructed covariantly from the fields and their derivatives. So we obtain the cohomological complex which we denote by $\Omega^*(\Gamma_1, \ldots, \Gamma_n)$. For example, if we have one field $\Gamma = (g_{ik})$ which is the Riemannian metric, then the form $\sum_{\alpha,\beta} R_{\alpha\beta} R_{\beta\alpha}$, where R is the curvature tensor is an example of a cocycle lying in $\Omega^4(\Gamma)$. Let us maintain that $\sum R_{\alpha\beta} R_{\beta\alpha}$ depending on g_{ik} and their derivatives is regarded as one cocycle – though, of course, it yields a four-form on any Riemannian manifold. The natural question arises, to find the cohomology of the complex $\Omega^*(\Gamma_1, \ldots, \Gamma_n)$. This cohomology is extremely important. Any cohomology class provides a non-trivial way of constructing covariantly (functorially) a closed form starting from the set of fields