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Quantum Affine Algebras and Holonomic Difference Equations

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Abstract. We derive new holonomic q-difference equations for the matrix coefficients of the products of intertwining operators for quantum affine algebra $U_q(\hat{\mathfrak{g}})$ representations of level k. We study the connection opertors between the solutions with different asymptotics and show that they are given by products of elliptic theta functions. We prove that the connection operators automatically provide elliptic solutions of Yang-Baxter equations in the "face" formulation for any type of Lie algebra $\mathfrak{g}$ and arbitrary finite-dimensional representations of $U_q(\hat{\mathfrak{g}})$. We conjecture that these solutions of the Yang-Baxter equations cover all elliptic solutions known in the contexts of IRF models of statistical mechanics. We also conjecture that in a special limit when $q \to 1$ these solutions degenerate again into $U_{q'}(\hat{\mathfrak{g}})$ solutions with $q' = \exp\left(\frac{2\pi i}{k+g}\right)$. We also study the simplest examples of solutions of our holonomic difference equations associated to $U_q(\widehat{\mathfrak{sl}(2)})$ and find their expressions in terms of basic (or $q-$)-hypergeometric series. In the special case of spin $-\frac{1}{2}$ representations, we demonstrate that the connection matrix yields a famous Baxter solution of the Yang-Baxter equation corresponding to the solid-on-solid model of statistical mechanics.

1. Introduction

The recent development in mathematics and physics related to conformal field theory [BPZ, FS, S, MS] and quantum groups [Kr, Dr1, J2] is a result of an astonishing interplay between various ideas of both sciences (see [0] for a partial bibliography). Mathematical roots of these theories lie in the representation theory of infinite dimensional Lie algebras and groups, algebraic geometry and Hamiltonian mechanics. The physical intuition arises from quantum field theory in two dimensions, integrable models in statistical mechanics and string theory. For mathematicians conformal field theory is a representation of certain geometric categories of Riemann Surfaces [S] or a regular representation of a "Lie algebra depending on a parameter" (vertex operator algebra) [FLM, MS]. For physicists, it is first of all the theory that characterizes the