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Nondegenerate Curves on S^2 and Orbit Classification of the Zamolodchikov Algebra

B. A. Khesin[1,*] **and B. Z. Shapiro**[2]

[1] Department of Mathematics, University of California, Berkeley, CA 94720, USA
[2] Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA

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Abstract. The Zamolodchikov algebra is the next case after the Virasoro algebra in the natural hierarchy of the Poisson structures on linear differential equations. We describe here the complete classification of the symplectic leaves of this algebra. It turns out that each symplectic leaf is uniquely defined by the conjugacy class of the monodromy operator and one discrete (2- and 3-valued) invariant arising from the homotopy classes of nondegenerate curves.

1. Introduction

The Zamolodchikov algebra is the algebra generated by the coefficients of the third order linear differential equations on the circle with respect to the quadratic Poisson structure [Z]. There exists a hierarchy of Poisson algebras on linear differential operators of different order on the circle also called the $SL_n(\mathbb{R})$ ($GL_n(\mathbb{R})$)-Gelfand-Dikii algebras or generalized the KdV-structures [GD]. The first Poisson algebra in this series on second order differential equations (more precisely on the Hill's equations) coincides with the Virasoro algebra [Kh]. The classification of the Virasoro coadjoint orbits was obtained in different terms independently by Kuiper [Ku], Lazutkin and Pankratova [LP], Segal [S], Kirillov [Ki].

In the Virasoro case, the Poisson algebra is linear, while for differential operators of any higher order the corresponding structure is quadratic. In the paper [OK] the classification of symplectic leaves (or maximal symplectic submanifolds) of these Poisson brackets for arbitrary order operators was related to the homotopy classification of some special curves on spheres (or in projective spaces). Namely, an n^{th} order linear differential operator on the circle defines a nondegenerate quasiperiodic curve in S^{n-1} (the "projectivization" of its "solution curve," see Sect. 2). It turned out that two differential operators belong to the same symplectic leaf iff the corresponding curves are homotopically equivalent.

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