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Spectral Properties of a Class of Operators Associated with Conformal Maps in Two Dimensions

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Abstract. If f is a rational map of the Riemann sphere, define the transfer operator $\mathscr{L}$ by

$$\mathscr{L}\Phi(z) = \sum_{Z:fZ=z} g(Z)\Phi(Z).$$

Let also $\mathscr{B}$ be the Banach space of functions for which the second derivatives are measures. If $g \in \mathscr{B}$ and g satisfies a simple integrability condition (implying that g vanishes at critical points and multiple poles of f) then $\mathscr{L}$ is a bounded linear operator on $\mathscr{B}$. The essential spectral radius of $\mathscr{L}$ can be estimated and, under suitable conditions, proved to be strictly less than the spectral radius. Similar estimates for more general operators $\mathscr{L}$ are also obtained.

1. Assumptions and Generalities

Let X be a bounded open subset of $\mathbb{C}$. If the second derivatives in the sense of distributions of $\varphi:\mathbb{C}\mapsto\mathbb{C}$ are bounded measures, we write (using a "functional" notation for measures):

$$\operatorname{Var}\varphi = \int_{\mathbb{C}} dx\,dy\,|\partial^2\varphi|,$$

where $|\partial^2\varphi|$ denotes the norm of the 2×2 matrix of second derivatives (i.e. the norm bilinear of forms on $\mathbb{C}^2$, where $\mathbb{C}^2$ has the usual Hilbert norm). In particular

$$\operatorname{Var}\varphi \geqq \frac{1}{4}\int_{\mathbb{C}} dx\,dy\left[\left|\frac{\partial^2\varphi}{\partial x^2}\right| + 2\left|\frac{\partial^2\varphi}{\partial x\,\partial y}\right| + \left|\frac{\partial^2\varphi}{\partial y^2}\right|\right].$$

We define

$$\mathscr{B} = \{\varphi : \operatorname{Var}\varphi < \infty \text{ and } \varphi \text{ vanishes on } \mathbb{C}\backslash X\}.$$

Then $\mathscr{B}$ is a Banach space with respect to the norm Var (which has properties similar to the total variation for functions on $\mathbb{R}$, particularly that it behaves well