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Rigorous Diffusion Properties for the Sawtooth Map

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Abstract. We get a rigorous bound for the diffusion constant of the hamiltonian dynamical system generated by a sawtooth map on a cylinder. The momentum variable properly renormalized then behaves almost like a brownian motion in the limit of infinite coupling constants. The strategy of the proof is a rigorous reformulation of the Random Phase Approximation.

0. Introduction

In this paper we consider the area-preserving sawtooth map:

$$\begin{aligned} A_{n+1} &= A_n + Kg(\theta_n), \\ \theta_{n+1} &= \theta_n + A_{n+1}\,(\mathrm{mod}\, 2\pi), \end{aligned} \tag{0.1}$$

where $g(\theta)$ is a 2π-periodic piecewise continuous function of zero average. This map is a simple model for a certain number of physical situations: charged particles in magnetic fields, plasma confinement in nuclear fusion, stochastic ionization, etc.... In fact it describes a "kicked" rotor subject to a sequence of periodic impulses $g(\theta)$, or alternatively, since it can be written as a second order difference equation:

$$\theta_{n+1} - 2\theta_n + \theta_{n-1} = Kg(\theta_n)\,(\mathrm{mod}\, 2\pi) \tag{0.2}$$

it describes the motion of a particle receiving an impulse determined by a periodic one-dimensional potential.

A lot of analytical and numerical work has been devoted to the study of (0.1) for smooth g's, the most important example being $g(\theta) = \sin\theta$, namely the Chirikov–Taylor "standard map" [1, 2]. While at small values of K the dynamics is regular (stability of KAM tori [3–8]), it has been observed since a long time ago that for K large, the solutions of (0.1) admit a diffusive behavior in the phase space $(A, \theta) \in \mathbb{R} \times \mathbb{T}$. The standard argument in this respect is the so-called *random phase*

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