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Level-Rank Duality of WZW Models in Conformal Field Theory

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Abstract. We consider the decomposition of the conformal blocks under the conformal embeddings. The case $\widehat{gl}(lr)_1 \supset \widehat{sl}(l)_r \times \widehat{sl}(r)_l \times \hat{a}$ ($\hat{a}$ is an affine extension of the abelian subalgebra of the central elements of $gl(lr)$) is studied in detail. The reciprocal decompositions of $\widehat{gl}(lr)_1$-modules induce a pairing between the spaces of conformal blocks of $\widehat{sl}(l)_r$ and $\widehat{sl}(r)_l$ Wess-Zumino-Witten models on the Riemann sphere. The completeness of the pairing is shown. Hence it defines a *duality* between two spaces.

1. Introduction

The curiosity about the affine Lie algebras pair $\widehat{sl}(l)_r$ and $\widehat{sl}(r)_l$ have appeared in several contexts. In [F], it was found that there is a mutually commutative embedding of affine Lie algebras

$$\widehat{sl}(l)_r \times \widehat{sl}(r)_l \subset \widehat{gl}(lr)_1. \tag{1.1}$$

Remarkably, mutually transposed pairs of representations $Y \otimes {}^tY$ of $\widehat{sl}(l)_r \times \widehat{sl}(r)_l$ appeared in the reciprocal decomposition of irreducible representations of $\widehat{gl}(lr)_1$ [F], and the decomposition formula for the character of $\widehat{gl}(lr)_1$ [JM]. The branching rule of the embedding (1.1) was studied in [H, ABI] in detail.

For any reductive affine Lie algebra, one can associate a Wess-Zumino-Witten (WZW) model [Wi] using the Sugawara construction [KZ, GWi]. Recently, two direct connections between $\widehat{sl}(l)_r$ and $\widehat{sl}(r)_l$ WZW models were found: The first one is an equality between the fusion rules of the both models proved in [KN, GWe] and partially proved in [SA]:

$$N_{Y_1Y_2}^{Y_3\ \widehat{sl}(l)_r} = N_{{}^tY_1{}^tY_2}^{{}^tY_3\ \widehat{sl}(l)_l}, \tag{1.2}$$

where the Young diagrams Y_i representing the primary fields, their sizes $|Y_i|$ are related as $|Y_1| + |Y_2| = |Y_3|$ and ${}^tY$ is the transposition of Y. The second one is strong evidence of the existence of a duality relation between the conformal blocks of 4-point functions of both models [NS].