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Dual Polygonal Billiards and Necklace Dynamics

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Abstract. We study the orbits of the dual billiard map about a polygonal table using the technique of necklace dynamics. Our main result is that for a certain class of tables, called the quasi-rational polygons, the dual billiard orbits are bounded. This implies that for the subset of rational tables (i.e. polygons with rational vertices) the dual billiard orbits are periodic.

1. Introduction

Let P be a closed bounded domain in $\mathbf{R}^2$ with a C^1 boundary and set $E=\mathbf{R}^2\backslash P$. If P is strictly convex, the *dual billiard* $T: E \to E$ is defined as follows. For any point $o \in E$ there are two rays R and R' emanating from o and tangent to P, where the observer looking at P from o sees R on the left and R' on the right of P. Let A and A' be the points of tangency. For any point $v \in \mathbf{R}^2$ denote by r_v the Euclidean reflection about v. Then $T(o)=r_A(o)$. The mapping T is continuous, preserves the Lebesgue measure and invertible with $T^{-1}(o)=r_{A'}(o)$.

If P is not strictly convex (for instance, P is a *convex polygon*) the dual billiard mapping T is defined the same way but not on all of E (Fig. 1). Denote by σ_1 the union of straight lines through the sides of P. Then both T and T^{-1} are defined on $E\backslash\sigma_1$ and $\sigma_1 \cap E$ is the union of singular sets of T and T^{-1}. By induction on $n \geqq 1$ we define σ_n, a finite union of straight lines, where T^k, $-n \leqq k \leqq n$, are well defined on $E\backslash\sigma_n$. The *singular set* $\Sigma=\bigcup_{n=1}^{\infty}\sigma_n$ is a countable union of straight lines, and for $x \in E\backslash\Sigma$ (regular points) the infinite orbits $\{T^n x: -\infty < n < \infty\}$ are defined. The theme of this work is the **orbit behavior for dual polygonal billiards**. In particular, can they be unbounded? If P is not a polygon but is bounded by a C^7-curve of positive curvature, all of the orbits are bounded [M1, D]. The proof is based on the

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