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Discrete Painlevé Equations and their Appearance in Quantum Gravity

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Abstract. We discuss an algorithmic approach for both deriving discrete analogues of Painlevé equations as well as using such equations to characterize "similarity" reductions of spatially discrete integrable evolution equations. As a concrete example we show that a discrete analogue of Painlevé I can be used to characterize "similarity" solutions of the Kac–Moerbeke equation. It turns out that these similarity solutions also satisfy a special case of Painlevé IV equation. In addition we discuss a methodology for obtaining the relevant continuous limits not only at the level of equations but also at the level of solutions. As an example we use the WKB method in the presence of two turning points of the third order to parametrize (at the continuous limit) the solution of Painlevé I in terms of the solution of discrete Painlevé I. Finally we show that these results are useful for investigating the partition function of the matrix model in 2D quantum gravity associated with the measure $\exp[-t_1z^2-t_2z^4-t_3z^6]$.

1. Introduction

Painlevé and Gambier [1], at the turn of the century, classified second order ODE's linear in the second derivative, whose solutions are free from movable branch points and essential singularities [2]. They found that, within a Möbius transformation, there exist fifty such equations. These equations can either be integrated in terms of known functions or can be reduced to one of six distinguished equations, called the six Painlevé transcendents. It is quite remarkable that although the Painlevé equations were introduced from purely mathematical considerations, they began, since the early 70's, appearing in several physical applications [3–4]. The appearance of these equations in physics (in particular in connection with the Ising model) and the realization that they are also closely related to integrable PDE's

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