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A Mathematical Approach to the Effective Hamiltonian in Perturbed Periodic Problems

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Abstract. We describe a rigorous mathematical reduction of the spectral study for a class of periodic problems with perturbations which gives a justification of the method of effective Hamiltonians in solid state physics. We study the partial differential operators of the form $P = P(hy, y, D_y + A(hy))$ on $\mathbb{R}^n$ (when $h > 0$ is small enough), where $P(x, y, \eta)$ is elliptic, periodic in y with respect to some lattice Γ, and admits smooth bounded coefficients in (x, y). $A(x)$ is a magnetic potential with bounded derivatives. We show that the spectral study of P near any fixed energy level can be reduced to the study of a finite system of h-pseudodifferential operators $\mathscr{E}(x, hD_x, h)$, acting on some Hilbert space depending on Γ. We then apply it to the study of the Schrödinger operator when the electric potential is periodic, and to some quasiperiodic potentials with vanishing magnetic field.

Introduction

The purpose of this paper is to give a rigorous mathematical treatment of an approximation widely used in solid state physics, namely the method of the effective Hamiltonian.

Let us briefly describe the essential ideas of this method: a typical problem to which this approximation is applied is the motion of an electron in a periodic crystal with a small external magnetic field. This problem is described by the following Hamiltonian:

$$H = \sum_1^3 (D_{y_j} + A_j(hy))^2 + V(y), \tag{0.1}$$

where V is a real potential, Γ-periodic for a lattice Γ in $\mathbb{R}^3$ describing the periodic crystal, and $A(x)$ is a function from $\mathbb{R}^3$ into $\mathbb{R}^{3*}$ (in other words a 1-form), which