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Adiabatic Limits of the η-Invariants The Odd-Dimensional Atiyah-Patodi-Singer Problem

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Abstract. We study the η-invariant of boundary value problems of Atiyah-Patodi-Singer type. We prove the formula for the spectral flow of the families over S^1. Assuming a product structure in a collar neighbourhood of the boundary, we show that the η-invariant behaves the same way as on a closed manifold. We also study the "adiabatic" limit of the η-invariant. In fact, we present a general method for the calculation of the "adiabatic" limits of the spectral invariants. In nice cases we are able to split them into a contribution from the interior, one from the cylinder, and an error term. Then we show that the error term disappears with the increasing length of the cylinder.

0. Introduction

Let $A:C^\infty(X;V)\to C^\infty(X;V)$ denote a generalized Dirac operator on an odd-dimensional manifold X with boundary Y, and $g:X\to U(N)$ denote a unitary gauge transformation equal to the identity on a certain neighbourhood of Y. We define the operators $D_0=A\otimes \mathrm{Id}_{\mathbb{C}^N}$ and $D_1=(\mathrm{Id}_V\otimes g)D_0(\mathrm{Id}_V\otimes g)^{-1}$ and the family $\{D_r=rD_1+(1-r)D_0\}$. We fix a self-adjoint boundary condition P for all D_r and as a result we obtain a self-adjoint Fredholm operator $(D_r)_P$. In fact, reduced modulo unitary equivalence $\{(D_r)_P\}_{r\in[0,1]}$ provides us with a family of self-adjoint Fredholm operators over the circle S^1.

The spectral flow is the only homotopy invariant of such families. It is the difference between the number of eigenvalues which change sign from $-$ to $+$ when r goes from 0 to 1 and the number of eigenvalues which change sign from $+$ to $-$. We want to compute this invariant. In the case of a closed manifold we use η-invariant. Let $\{B_r\}$ denote a family of self-adjoint elliptic operators of positive order on a closed manifold. The topological formula for the spectral flow is the result of the following equality:

$$\mathrm{sf}\{B_r\}=\int_0^1 d/dr(\eta_{B_r}(0))dr\,, \tag{0.1}$$