Commun. Math. Phys. 141, 559–576 (1991)

Communications in Mathematical Physics

Convergence of Nelson Diffusions

Gianfausto Dell'Antonio[1,2] **and Andrea Posilicano**[2]

[1] Dipartimento di Matematica, Università di Roma I, I-00185 Roma, Italy
[2] S.I.S.S.A., I-34014 Trieste, Italy

Received November 26, 1990; in revised form May 3, 1991

Abstract. Let ψ_t, ψ_t^n, $n \geqq 1$, be solutions of Schrödinger equations with potentials form-bounded by $-\frac{1}{2}\Delta$ and initial data in $H^1(\mathbb{R}^d)$. Let P, P^n, $n \geqq 1$, be the probability measures on the path space $\Omega = C(\mathbb{R}_+, \mathbb{R}^d)$ given by the corresponding Nelson diffusions. We show that if $\{\psi_t^n\}_{n \geqq 1}$ converges to ψ_t in $H^1(\mathbb{R}^d)$, uniformly in t over compact intervals, then $\{P^n_{|\mathscr{F}_t}\}_{n \geqq 1}$ converges to $P_{|\mathscr{F}_t}$ in total variation $\forall t \geqq 0$. Moreover, if the potentials are in the Kato class K_d, we show that the above result follows from H^1-convergence of initial data, and K_d-convergence of potentials.

1. Introduction

Stochastic Quantization is an algorithm which permits to associate a diffusion process to a solution of the Schrödinger equation in such a way that the density of the process corresponds to the usual density of Quantum Mechanics (see [N] for a thorough introduction to the subject). An unpleasant characteristic of these diffusion processes is that their drift coefficients are too singular to be handled by the traditional approaches. The problem of the existence of the stochastic processes of Stochastic Mechanics was resolved by Carlen for potentials form-bounded by $-\frac{1}{2}\Delta$ and initial data in $H^1(\mathbb{R}^d)$ (see [C1, C2, C3], and Sect. 2). His existence theorem provides us a Borel probability measure on $\Omega = C(\mathbb{R}_+, \mathbb{R}^d)$, the space of the physical trajectories of the particles, such that the stochastic process $X_t(\gamma) := \gamma(t)$ is a Markov process with density $|\psi_t|^2$, and is a weak solution of the stochastic differential equation

$$X_t = X_0 + \int_0^t b(s, X_s) ds + B_t$$

with $b_t = (\Re + \Im) \nabla \log \psi_t$, as required by the stochastic quantization procedure. Carlen's theorem provides a map from the space of solutions of the Schrödinger equation to the space $\mathscr{M}_1(\Omega)$ of probability measures on Ω; it is then natural to consider the following continuity problem: let $\{\psi_t^n\}_{n \geqq 1}$ be a sequence of solutions