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On Rational Solutions of Yang–Baxter Equations. Maximal Orders in Loop Algebra

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Abstract. In 1982 Belavin and Drinfeld listed all elliptic and trigonometric solutions $X(u,v)$ of the classical Yang–Baxter equation (CYBE), where X takes values in a simple complex Lie algebra $\mathfrak{g}$, and left the classification problem of the rational one open. In 1984 Drinfeld conjectured that if a rational solution is equivalent to a solution of the form $X(u,v)=C_2/(u-v)+r(u,v)$, where C_2 is the quadratic Casimir element and r is a polynomial in u,v, then $\deg_u r=\deg_v r\leqq 1$. In another paper I proved this conjecture for $\mathfrak{g}=\mathfrak{sl}(n)$ and reduced the problem of listing "nontrivial" (i.e. nonequivalent to $C_2/(u-v)$) solutions of CYBE to classification of quasi-Frobenius subalgebras of $\mathfrak{g}$. They, in turn, are related with the so-called maximal orders in the loop algebra of $\mathfrak{g}$ corresponding to the vertices of the extended Dynkin diagram $D^e(\mathfrak{g})$. In this paper I give an algorithm which enables one to list all solutions and illustrate it with solutions corresponding to vertices of $D^e(\mathfrak{g})$ with coefficient 2 or 3. In particular I will find all solutions for $\mathfrak{g}=\mathfrak{o}(5)$ and some solutions for $\mathfrak{g}=\mathfrak{o}(7)$, $\mathfrak{o}(10)$, $\mathfrak{o}(14)$ and $\mathfrak{g}_2$.

Introduction

This paper is a continuation of refs. [11–15]. I will recall, however, some of the notations and the main idea. In this paper I will explain how rational solutions of the classical Yang–Baxter equation (CYBE) for a simple complex Lie algebra $\mathfrak{g}$ correspond to the extended Dynkin diagram $D^e(\mathfrak{g})$. An announcement of the results of this paper had been delivered at the International Algebraic Conference in Novosibirsk, 1989 [13, 14].

0.1. Formulation of the Problem. We will consider functions $X:\mathbb{C}^2\to\mathfrak{g}\otimes\mathfrak{g}$ such that

$$[X^{12}(u_1,u_2),X^{13}(u_1,u_3)]+[X^{12}(u_1,u_2),X^{23}(u_2,u_3)]+[X^{13}(u_1,u_3),X^{23}(u_2,u_3)]=0,$$
$$X^{12}(u,v)=-X^{21}(v,u),\quad \text{(CYBE)}$$

and a solution will be called *rational* if it is of the form $X=C_2/(u-v)+r(u,v)$, where $r(u,v)\in\mathfrak{g}[u]\otimes\mathfrak{g}[v]$, cf. refs. [2, 3].