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Julia Sets and Complex Singularities in Hierarchical Ising Models

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Abstract. We study the analytical continuation in the complex plane of free energy of the Ising model on diamond-like hierarchical lattices. It is known [12, 13] that the singularities of free energy of this model lie on the Julia set of some rational endomorphism f related to the action of the Migdal–Kadanoff renorm-group. We study the asymptotics of free energy when temperature goes along hyperbolic geodesics to the boundary of an attractive basin of f. We prove that for almost all (with respect to the harmonic measure) geodesics the complex critical exponent is common, and compute it.

1. Introduction

The purpose of this article is to analyse complex singularities in temperature of the free energy $\mathscr{F}$ in the Ising model on diamond-like hierarchical lattices. According to the traditional point of view a phase transition manifests itself as a singularity of $\mathscr{F}$ as a function of thermodynamic parameters (like temperature and external magnetic field). From this point of view the theory of phase transitions should describe the domain of analyticity of $\mathscr{F}$ and the type of its singularities at points of phase transition (see [1], where diverse approaches to the first of these problems are discussed).

Since $\mathscr{F}$ is real analytic outside of points of phase transition, it can be continued into complex space with respect to the thermodynamic parameters. Description of its complex singularities is of great interest for the theory of phase transitions because it determines analytic properties of the thermodynamic function.

The celebrated Lee–Yang theory (see [2]) gives a realisation of this approach describing the singularities of the analytic continuation of the free energy in the ferromagnetic Ising model with respect to the external magnetic field. It proves that the zeroes of the grand partition function in the ferromagnetic Ising model lie on the imaginary axis, and hence complex singularities of the free energy lie on the imaginary axis as well. An important problem stated in [2] is to study the