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# String Structures and the Path Fibration of a Group

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**Abstract.** We use the realisation of the universal bundle for the loop group as the path fibration of the group to investigate the string class, that is the obstruction to a loop group bundle lifting to a Kac–Moody group bundle. In the case that the loop group bundle is constructed by taking loops into a principal bundle we show that the classifying map is the holonomy around loops and give an explicit formula for the string class relating it to the Pontrjagin class of the principal bundle.

## 1. Introduction

Consider a principal bundle $\pi: P \to M$, with structure group a Lie group $G$, a basepoint $m_0$ chosen in $M$ and a basepoint $p_0 \in \pi^{-1}(m_0)$ chosen in the fibre of $P$ over $m_0$. For any space $X$ with basepoint $x_0$ we denote by $\Omega X$ the space of smooth loops in $X$ based at $x_0$. To be precise, by a smooth loop we mean a smooth path whose endpoints are coincident and for a based loop the endpoints coincide at the basepoint. We take the basepoint of $G$ to be the identity $e$. With these definitions we can construct another principal bundle $p: \Omega P \to \Omega M$ with structure group $\Omega G$. The map $p$ sends a loop $\gamma(\theta)$ in $P$ to the loop $\pi(\gamma(\theta))$ in $M$. We shall call such a principal $\Omega G$ bundle a *loop bundle.*

If $G$ is a compact Lie group then the loop group $\Omega G$ has a well-known central extension (see for instance Pressley and Segal 1986)

$$1 \to U(1) \to \widehat{\Omega G} \to \Omega G \to 1, \tag{1.1}$$

and therefore for any principal $\widehat{\Omega G}$ bundle there is an induced $\Omega G$ bundle obtained, for instance, by composing transition functions for $\widehat{\Omega G}$ with the homomorphism $\widehat{\Omega G} \to \Omega G$. Given a principal $\Omega G$ bundle then, it is natural to ask when it is induced in this way from an $\widehat{\Omega G}$ bundle. In such a case we say that the $\Omega G$ bundle *lifts* to an $\widehat{\Omega G}$ bundle. An analogous problem arises for the group $SO(n)$ and the