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Computer Calculation of Witten's 3-Manifold Invariant*

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Abstract. Witten's $2+1$ dimensional Chern-Simons theory is exactly solvable. We compute the partition function, a topological invariant of 3-manifolds, on generalized Seifert spaces. Thus we test the path integral using the theory of 3-manifolds. In particular, we compare the exact solution with the asymptotic formula predicted by perturbation theory. We conclude that this path integral works as advertised and gives an effective topological invariant.

Quantum Field Theory is rapidly emerging as the unifying principle behind new topological invariants in low dimensions. Ed Witten led this development with his introduction of *topological* quantum field theories. From many points of view the most accessible of these quantum theories is the $2+1$ dimensional Chern-Simons theory [W]. There is a beautiful corresponding classical theory [F1] defined for each class in $H^4(BG)$, where G is a compact Lie group and BG its classifying space. In case G is finite the quantization is straightforward and illuminating [DW, FQ], though the topological invariants arising from the quantum theory are trivial. On the other hand the quantization of the theory for continuous groups proceeds via the Feynman path integral, which is not (yet) a mathematically rigorous procedure. Nevertheless, this quantum theory (presuming it exists) reproduces the Jones polynomial of links in S^3 and generates new invariants of links in arbitrary closed oriented 3-manifolds. Some of Witten's assertions in [W] now have mathematical proofs independent of path integral arguments. We give evidence of a different kind for the validity of the quantum theory – computer calculations.

In this paper we restrict our attention to the simplest continuous group $G=SU(2)$. Then $H^4(BG)\cong\mathbb{Z}$, so there is an $SU(2)$ theory for each integer k,

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