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Maxwell's Equations in Divergence Form for General Media with Applications to MHD*

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Abstract. Maxwell's equations in media with general constitutive relations are reformulated in covariant form as a system of divergence equations without constraints. Our reformulation enables us to express general electro-magneto-fluid problems as hyperbolic systems in divergence form. We illustrate this method on the MHD problem. In the absence of constraints, a general representation is derived for the characteristic form for first-order systems of quasi-linear partial differential equations in vector fields and scalars. Using this covariant formulation of characteristics, we find that the principle of covariance imposes a very rigid structure on the infinitesimally small amplitude waves in MHD. To demonstrate the power of the reformulation, we study numerically ultra-relativistic wave breaking using the divergence formulation of MHD.

Introduction

Maxwell's equations appear in a wide variety of problem settings in general relativity. We will consider them as they appear in general relativistic formulations of electro-magneto-fluid problems. They appear in their natural form as an underdetermined system of divergence equations. Lichnerowicz [5] showed that implementation of constitutive relations of a particular medium yields a pair of scalar constraints. Thus, electromagnetic fields in general media are determined completely by a mixed partial differential-algebraic system of equations.

Numerical treatment of electro-magneto-fluid problems by standard methods requires these problems to be formulated as a system consisting purely of partial differential equations with no constraints. Of course, the constraints as they appear in Lichnerowicz's formulation are avoided when taking the electromagnetic field variables as 3-vectors (cf. [12, 18]). The electromagnetic fields in general

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