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Condensation of a One-Dimensional Lattice Gas

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Abstract. We consider a one-dimensional lattice gas in the canonical ensemble with interaction energy $1/r^{\alpha}$, $1 < \alpha \leq 2$. Using an energy-entropy argument we show that the gas condenses at sufficiently low temperatures meaning that the gas has a non-uniform density in the thermodynamic limit.

0. Introduction

Consider a one-dimensional lattice gas in the canonical ensemble, i.e. the number of positions in $[1, L] \cap Z$ occupied by particles is a fixed number N. Let the interaction energy between two particles at distance r be $-1/r^{\alpha}$, $1 < \alpha \leq 2$. We will show that, with appropriate boundary conditions, this lattice gas condenses at sufficiently low temperatures, meaning that the gas has a non-uniform density in the thermodynamic limit $N, L \to \infty, N/L \to \varrho, 0 < \varrho < 1$.

The proof is an energy-entropy argument which makes rigorous the following heuristic argument, see Landau-Lifshitz [6] abd Thouless [9]. Consider configurations that are not condensed already, say those that are approximately uniform. Suppose that we can partition these configurations into blocks $A_1, B_1, A_2, \ldots, A_k, B_k$, where almost every position in $A_1, \ldots, A_k$ is occupied and almost every position in $B_1, \ldots, B_k$ is empty. Also assume that all these blocks have length $\geq d$. Rearrange the blocks so that the condensed configuration $A_1 A_2 \ldots A_k B_1 \ldots B_k$ is obtained. The change in energy is $\Delta E \sim -CkE_{\alpha}(d)$, where $E_{\alpha}(d) = d^{2-\alpha}$ if $1 < \alpha < 2$, $= \log d$ if $\alpha = 2$ and $\leq$ const. if $\alpha > 2$. The change in entropy due to many configurations being mapped to the same condensed configuration is roughly $\Delta S \sim -\log \binom{L}{k} \sim -k \log(L/k)$. The free energy thus changes by $\Delta F = \Delta E - \beta^{-1} \Delta S \sim k(-CE_{\alpha}(d) + \beta^{-1} \log(L/k))$. If $\alpha > 2$ and $k = o(N)$,

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