Commun. Math. Phys. 141, 1–8 (1991)

Communications in Mathematical Physics

Metamorphoses: Sudden Jumps in Basin Boundaries*

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Received August 14, 1986; in revised form April 15, 1990

Abstract. In some invertible maps of the plane that depend on a parameter, boundaries of basins of attraction are extremely sensitive to small changes in the parameter. A basin boundary can jump suddenly, and, as it does, change from being smooth to fractal. Such changes are called *basin boundary metamorphoses*. We prove (under certain non-degeneracy assumptions) that a metamorphosis occurs when the stable and unstable manifolds of a periodic saddle on the boundary undergo a homoclinic tangency.

Dynamical systems in the plane can have many coexisting attractors. In order to be able to predict long-term or asymptotic behavior in such systems, it is important to be able to recognize to which attractor (final state) a given trajectory will tend. The set of initial conditions whose trajectories are asymptotic to a particular attractor is called the *basin of attraction* of that attractor. In some systems that depend on a parameter, it has been observed that the boundaries of these basins are extremely sensitive to small changes in the parameter. Not only can a boundary jump suddenly, but it can also change from being smooth to being fractal. These changes, called *boundary metamorphoses*, are studied at length in [GOY]. In this paper we prove a theorem, originally stated in [GOY], which characterizes the jumps in basin boundaries.

The Hénon map $f(x,y)=(A-x^2-Jy,x)$ provides an example of this phenomenon. We fix $J=0.3$ and vary A, resulting in a one-parameter, invertible map of the plane. The Jacobian of f is J; hence, f is area contracting for all A. We will be looking specifically at the boundary of the basin of attraction of infinity. (The basin of infinity is the set of all points (x,y) such that $|f^n(x,y)|\to\infty$ as $n\to\infty$.) Figures 1a and 1b show the basin of infinity in black for $A=1.314$ and $A=1.320$, respectively. In Fig. 1b we see that the basin of infinity contains points which were previously (at $A=1.314$) well within the white region. This new set of black points has not

* This research was supported in part by grants and contracts from the Defense Advanced Research Projects Agency, the Consiglio Nazionale delle Ricerche (Comitato per le Matematiche), and the National Science Foundation