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Unitary Evolutions and Horizontal Lifts in Quantum Stochastic Calculus

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Abstract. Unitarity is proved for a class of solutions of quantum stochastic differential equations with unbounded coefficients. The resulting processes are then used to construct algebraic quantum diffusions. Applications include an existence proof for a class of diffusions on the non-commutative two-torus and a geometric interpretation for diffusions driven by the classical Poisson process.

1. Introduction

Let $\mathscr{h}_0$ be a complex, separable Hilbert space and Γ be boson Fock space over $L^2([0,\infty))$. A major role in the development of quantum stochastic calculus has been played by processes $U=(U(t), t \geqq 0)$ which arise as the solutions of the following linear stochastic differential equation on $\mathscr{h}_0 \otimes \Gamma$:

$$\begin{aligned} dU &= U(L_1 \otimes dA^\dagger + L_2 \otimes d\Lambda + L_3 \otimes dA + L_4 \otimes I\,dt), \\ U(0) &= I, \end{aligned} \tag{1.1}$$

where $L_j \in B(\mathscr{h}_0)$ $(1 \leqq j \leqq 4)$ and $A^\dagger, \Lambda$ and A are the processes of creation, conservation and annihilation in Γ (respectively) ([13, 18]). We note that the coefficients $L_j (1 \leqq j \leqq 4)$ do not here depend on time.

Of particular importance are the sub-class of solutions to (1.1) where the algebraic relations between the L_j's are such that U is unitary operator valued. These unitary processes have found many applications including the construction of Markov dilations of quantum dynamical semigroups ([13, 14]), the description of quantum processes with stationary and independent increments over graded *-bialgebras [1] and the representation of time-ordered stochastic product integrals [11].

To the extent that these unitary processes may be regarded as solutions of a "Schrödinger equation in the presence of noise" [18], it is clear that the case where the L_j's are no longer bounded will be of great practical interest. So far, however, although existence of solutions to (1.1) was established in [13] there has been no