Commun. Math. Phys. 137, 29–66 (1991)

Communications in Mathematical Physics

Quantization of Chern–Simons Gauge Theory with Complex Gauge Group

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Received January 8, 1990

Abstract. The canonical quantization of Chern–Simons gauge theory in $2+1$ dimensions is generalized from the case in which the gauge group is a compact Lie group G to the case in which the gauge group is a complex Lie group $G_{\mathbb{C}}$. Though the physical Hilbert spaces become infinite dimensional in the latter case, the quantization can be described as precisely as for compact gauge groups and using similar methods. The special case in which the gauge group is $SL(2,\mathbb{C})$ gives a description of $2+1$ dimensional quantum gravity with Lorentz signature and positive cosmological constant or with Euclidean signature and negative cosmological constant. While it is not clear whether there is a $1+1$ dimensional conformal field theory related to these $2+1$ dimensional models, there are natural, computable candidates for the central charge and the conformal blocks of such a hypothetical theory.

1. Introduction

In Yang–Mills theory, with a gauge group G, letting A denote a connection on a G bundle E over a space-time manifold M, the conventional action functional is

$$I = \frac{1}{4g^2}\int_M \mathrm{Tr}(F_{ij}F^{ij}). \tag{1.1}$$

Here "Tr" represents an invariant and non-degenerate quadratic form on the Lie algebra $\mathbf{g}$ of G, customarily denoted by $(a,b) = \mathrm{Tr}(ab)$, and $F = dA + A \wedge A$ is the Yang–Mills field strength.

In studying the theory associated with the Lagrangian (1.1), one usually requires that the quadratic form defined by "Tr" should be positive definite, but the reason for this requirement is not always stated accurately. If the quadratic form used in (1.1) is indefinite, then the kinetic energy of the gauge bosons is indefinite. It is

* Research supported in part by NSF Grant 86-20266 and NSF Waterman Grant 88-17521