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Continuity and Relative Hamiltonians

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Abstract. Let $(\omega_n)_{n \geqq 1}$ be a norm convergent sequence of normal states on a von Neumann algebra $\mathscr{A}$ with $\omega_n \to \omega$. Let $(k_n)_{n \geqq 1}$ be a strongly convergent sequence of self-adjoint elements of $\mathscr{A}$ with $k_n \to k$. It is shown that the sequence $(\omega_n^{k_n})_{n \geqq 1}$ of perturbed states converges in norm to ω^k. A related result holds for C^*-algebras. A counter-example is provided to show that it is not sufficient to assume weak convergence of $(\omega_n)_{n \geqq 1}$ even when $k_n = k$ for all n. However, conditions are given which, together with weak convergence, are sufficient. Relative entropy methods are used, and a relative entropy inequality is proved.

1. Continuity of ω^k in ω and in k

Given a faithful state ω on a von Neumann algebra $\mathscr{A}$ and a self-adjoint element $k \in \mathscr{A}$, Araki [1] defined, using perturbation theory and modular theory, a state denoted by ω^k. The motivation for this definition came from quantum statistical mechanics. If ω represents the equilibrium state of a physical system, then ω^k will represent the equilibrium state of the perturbed system in which the energy of each state σ has been increased by $\sigma(k)$. Araki's definition has proved useful for the analysis of stability properties for equilibrium states and in demonstrating the invariance of such states under given symmetry groups.

In [7], I have used the equivalent, but more direct, definition, that ω^k is the unique state maximizing the function $\sigma \mapsto \text{ent}_{\mathscr{A}}(\sigma|\omega) - \sigma(k)$, where $\text{ent}_{\mathscr{A}}(\sigma|\omega)$ is the relative entropy of σ with respect to ω. With this definition, it is possible to use relative entropy techniques to give alternative – and, in my view, simpler – proofs of the results in [1]. Also, the definition and many of the results can be extended to the case in which k is a lower-bounded self-adjoint operator affiliated with $\mathscr{A}$. In this paper, relative entropy techniques are used to prove a powerful continuity result for ω^k with k bounded. Background and full elucidation of the notation are given in [7]. I have used [7] wherever possible in this paper as a unified source of results about ω^k, but, of course, many of the results quoted were originally proved