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Propagating Fronts and the Center Manifold Theorem

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Abstract. We prove the existence of propagating front solutions for the Swift–Hohenberg equation (SH). Using the center manifold theorem we reduce the problem to a two dimensional system of ordinary differential equations. They describe stationary solutions and front solutions of the partial differential equation (SH). We identify the well-known "amplitude equation" as the lowest order approximation to the equation of motion on the center manifold.

1. Introduction

In this paper, we reconsider the existence problem for fronts in the Swift–Hohenberg equation, which was studied in [CE1, CE2]. This equation is of the form

$$\partial_t u(x,t) = (\alpha - (1+\partial_x^2)^2)u(x,t) - u^3(x,t). \tag{1.1}$$

It is known that for small positive α this equation has *stationary solutions* (i.e., time-independent solutions) which are periodic with period ω, for $|\omega|$ close to 1. If we define $\varepsilon > 0$ by

$$(\omega^2 - 1)^2 + \varepsilon^2 = \alpha,$$

then these solutions are of the form

$$u(x) = S(x) \approx \frac{2}{\sqrt{3}}\varepsilon\cos(\omega x).$$

Furthermore, in [CE1], *front solutions* for Eq. (1.1) were defined as solutions of the form

$$u(x,t) = W(x, x-ct) \tag{1.2}$$

with the boundary conditions at infinity

$$\lim_{y\to-\infty} W(x,y) = S(x), \qquad \lim_{y\to+\infty} W(x,y) = 0.$$