Commun. Math. Phys. 135, 201–216 (1990)

Communications in
Mathematical
Physics

Generalized Lax Pairs, the Modified Classical Yang-Baxter Equation, and Affine Geometry of Lie Groups

Martin Bordemann

Fakultät für Physik der Universität Freiburg, Hermann-Herder-Strasse 3,
W-7800 Freiburg i. Br., Federal Republic of Germany

Received May 2, 1990

Abstract. We generalize the usual Lax equation $\frac{d}{dt}L=[M,L]$ by $\frac{d}{dt}L=-\varrho(M)L$, where ϱ is an arbitrary representation of a Lie algebra $\mathfrak{g}$ (the values of M) in a representation space V (the values of L). The usual classical r-matrix programme for Hamiltonian integrable systems is generalized to r-matrices taking values in $\mathfrak{g}\otimes V$. The r-matrices are then considered as left invariant torsion-free covariant derivatives on a Lie group K (with Lie algebra V^*). The Classical Yang-Baxter Equation (CYBE) is equivalent to the flatness of K whereas the Modified CYBE implies that K is an affine locally symmetric space. An example is discussed.

1. Introduction

In the last years, the interest in integrable Hamiltonian systems has rapidly grown because of the relation to the Classical Yang-Baxter Equations (CYBE) which in turn point towards the Quantum Yang-Baxter Equations (QYBE) and Quantum Groups. In various papers, e.g. by Sklyanin (cf. [1, 2]), Belavin and Drinfel'd (cf. [3, 4]) and Semenov-Tyan-Shanskii (cf. [5]) classical r-matrices have been defined and discussed. Further algebraic properties and formulations of the CYBE and Poisson Lie groups using Schouten brackets are developed in the work of Kosmann-Schwarzbach and Magri (cf. [6]). In several articles (cf. [7, 8]) Babelon and Viallet studied the importance of classical r-matrices and the CYBE in the context of Hamiltonian systems defined by Lax pairs.

In this paper, an attempt is made to generalize the concept of Lax pairs by replacing the well-known Lax equation (cf. [9])

$$\frac{dL}{dt}=[L,M]$$

by

$$\frac{dL}{dt}=-\varrho(M)L,$$