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Variational Processes from the Weak Forward Equation

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Abstract. In this paper the author constructs Markov diffusion processes from a given system of Borel probability measures on a d-dimensional Euclidean space. He constructs a, so-called, variational process which does not always coincide with a Nelson process. He also discusses Schrödinger's problem in quantum mechanics.

0. Introduction

The theory of Markov processes has been developed by many authors (cf. Dynkin [10, 11]). But

(Q) Under what information can we assume that the real world is Markovian?

Of course, it depends on how we consider the real world. But it is better if we can assume that it is Markovian, because Markov property is a nice (kind of ideal) property. Before we state our problem, we mention that some notations are given in the end of this section.

In this paper we consider the following problem; let us fix $T > 0$. Assume that we are given the system of Borel probability measures $\{\rho(t, dx)\}_{0 \leqq t \leqq T}$ on $(\mathfrak{R}^d, B(\mathfrak{R}^d))$ which satisfies the following weak forward equation; for any $f \in C_b^{1,2}([0, T] \times \mathfrak{R}^d; \mathfrak{R})$ and any $0 \leqq s \leqq t \leqq T$,

$$\begin{aligned}
&\int_{\mathfrak{R}^d} f(t,x)\rho(t,dx) - \int_{\mathfrak{R}^d} f(s,x)\rho(s,dx) \\
&= \int_s^t \int_{\mathfrak{R}^d} \Big[\partial f(u,x)/\partial u + \left(\sum_{i,j=1}^d a^{ij}(u,x)\partial^2 f(u,x)/\partial x_i \partial x_j\right)\Big/ 2 \\
&\quad + \sum_{i=1}^d b^i(u,x)\partial f(u,x)/\partial x_i\Big]\rho(u,dx)du,
\end{aligned} \tag{0.1}$$