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Generalized Apollonian Packings

Daniel Bessis[1] and Stephen Demko[2]

[1] Service de Physique Théorique* de Saclay, F-91191 Gif-sur-Yvette Cedex, France
[2] School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332, USA

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Abstract. In this paper we generalize the classical two-dimensional Apollonian packing of circles to the case where the circles are no more tangent. We introduce two elements of $SL(2,\mathbb{C})$ as generators: R and T that are hyperbolic rotations of $\frac{2\pi}{3}$ and $\frac{2\pi}{N}$ $(N=2,3,4,\ldots)$, around two distinct points. The limit set of the discrete group generated by R and T provides, for $N=7,8,9,\ldots$ a generalization of the Apollonian packing (which is itself recovered for $N=\infty$). The values $N=2,3,4,5$ produce a very different result, giving rise to the rotation groups of the cube for $N=2$ and 4, and the icosahedron for $N=3$ and 5. For $N=6$ the group is no longer discrete. To further analyze this structure for $N\geqq 7$, we move to the Minkowski space in which the group acts on a one sheeted hyperboloid. The circles are now represented by points on this variety and generate a crystal on it.

I. Introduction

In a classical construction of an Appolonian packing, one starts with a curvilinear triangle and constructs the inscribed circle, thus creating three curvilinear triangles out of the original one. This process is then repeated with each of the resulting curvilinear triangles and their descendants. The method for producing the inscribed circles can be realized with inversions [1] or Möbius maps [2].

In this latter case, one sees that the Apollonian packing is the limit set of a discrete subgroup of $SL(2,\mathbb{C})$. We present here a technique for generating non-tangential disk packings as limit sets of discrete groups which include as a special case the Apollonian packing. Extension to higher dimensional sphere packings will be presented elsewhere.

Disk and sphere packings are natural models for porous media [3,4] and a

* Laboratoire de l'Institut de Recherche Fondamentale du Commissariat à l'Energie Atomique