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A Lie Algebraic Setting for Miura Maps Related to an Energy Dependent Linear Problem

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Abstract. We study the affine semi-direct product, $\mathrm{Vect}(S^1) \ltimes C^\infty(S^1, \mathbb{R}) \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ and show how the coadjoint action on its extended dual leads immediately to a generalisation of the Miura map and also to the existence of several compatible Hamiltonian structures.

Introduction

With a view to developing a clearer understanding of the underlying structures responsible for the exemplary behaviour of integrable systems, it is worthwhile trying to place results within the context of the now very familiar scenario of coadjoint actions of Lie groups. In this paper we are interested in looking at the so-called "energy-dependent" linear problem which is found to generate Miura maps (i.e. noninvertible Poisson maps) rather like the original one relating the MKdV and KdV equations, see [3].

We follow the approach explained in a paper of Reyman and Semenov-Tian-Shansky [1]. They showed how a family of r-brackets could be defined on an affine Lie algebra – these immediately giving rise to a family of compatible Poisson brackets on the dual. These authors further showed in [2] how this approach could be used to explain the existence of a family of Poisson structures associated with the problem

$$(e\partial^2 + u)\psi = 0$$

with e and u polynomials in the spectral parameter. Their analysis essentially amounts to the recognition that the relevant Lie algebra to look at is the Virasoro Lie algebra. In our case we consider an enlargement of the Virasoro algebra by means of the semidirect product.

Energy dependent Miura maps were investigated by a more direct method by Antonowicz and Fordy [3]. It should be emphasized that the structures described