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# Crystalizing the $q$-Analogue of Universal Enveloping Algebras

**Masaki Kashiwara**

RIMS, Kyoto University, Kyoto 606, Japan

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**Abstract.** For an irreducible representation of the $q$-analogue of a universal enveloping algebra, one can find a canonical base at $q=0$, named crystal base (conjectured in a general case and proven for $A_n, B_n, C_n$ and $D_n$). The crystal base has a structure of a colored oriented graph, named crystal graph. The crystal base of the tensor product (respectively the direct sum) is the tensor product (respectively the union) of the crystal base. The crystal graph of the tensor product is also explicitly described. This gives a combinatorial description of the decomposition of the tensor product into irreducible components.

## 0. Introduction

The $q$-analogue of a universal enveloping algebra introduced by Drinfeld [2] and Jimbo [3] is a deformation of the universal enveloping algebra at $q=1$. Since $q=0$ corresponds to the absolute temperature zero in the lattice model defined by the $R$-matrix, we can expect that the $q$-analogue has a simple structure in that case. Some indications have been already observed in Date–Jimbo–Miwa [1], where the Gelfand–Tsetlin bases become monomes in the tensor algebra of the fundamental representation when $q=0$. In this note, we shall clarify this phenomenon. For an irreducible representation of the $q$-analogue, we can find a canonical base at $q=0$, named crystal base (conjectured in a general case and proven in $A_n, B_n, C_n, D_n$). The crystal base of the tensor product is the tensor product of the crystal bases. Moreover the crystal base has a structure of colored oriented graph (called crystal graph). The crystal graph of the tensor product is explicitly described.

We shall state our results more precisely. We introduce operators $\tilde{e}_i$ and $\tilde{f}_i$ by modifying the simple root vectors $e_i$ and $f_i$ of the $q$-analogue $U_q$ (see Sect. 3). Let $M$ be an integrable representation of $U_q$ defined over $\mathbf{Q}(q)$. We consider a pair $(L, B)$ of a lattice $L$ of $M$ defined over the ring of rational functions in $q$ regular