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Zeroes of Chromatic Polynomials: A New Approach to Beraha Conjecture Using Quantum Groups

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Abstract. The number of colourings of a graph G with Q or fewer colors is a polynomial in Q known as the chromatic polynomial $P_G(Q)$. It coïncides with the partition function $\mathscr{Z}_G$ of the Q state Potts model on G at zero temperature and in the antiferromagnetic regime $e^K=0$. In the planar case, the Beraha conjecture particularizes the numbers $B_n=4\cos^2\frac{\pi}{n}$ as possible accumulation points of real zeroes of P_G in the infinite graph limit. We suggest in this work an approach based on recent developments of quantum groups to handle this conjecture. For the square, triangular and honeycomb lattices and systems wrapped on a cylinder $l\times t$, we first exhibit in the (Q, e^K) Potts parameter space a critical line, where $\mathscr{Z}_G(Q, e^K)$ has real zeroes converging to and only to the B_n's as $l, t\to\infty$. The analysis is based on the vertex representation of the Q state Potts model, quantum algebra $U_q sl(2)$ properties for q a root of unity, and conformal invariance. $U_q sl(2)$ symmetry is present for any e^K, including the chromatic polynomial case $e^K=0$. Using an additional hypothesis on the eigenvalues structure and knowledge of the Potts parameter space, we then argue that for $P_G(Q)$, real zeros occur and converge to B_n's, $2\leqq n\leqq n_0$ only, where n_0 depends on the lattice. Extensions to other kinds of graphs and size dependence of the zeros are discussed. Finally physical applications are also mentioned.

1. Introduction

Consider a graph G, i.e. a set of points and edges joining pairs of points [1]. The number of ways of colouring the points of G with $Q\in\mathbb{N}$ or fewer colours, no two adjacent points having the same colour, is a polynomial in Q known as the chromatic polynomial: $P_G(Q)$ [1].

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[1] We suppose that G has no loop, otherwise P_G would be trivially 0 for any Q. Moreover we can restrict to graphs with no parallel edges, since their presence does not affect the chromatic problem