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Theory of Ordered Spaces

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Abstract. This is the first of a planned series of investigations on the theory of ordered spaces based upon four axioms. Two of these, the order (I.1.1) and the local structure (II.5.1) axioms provide the structure of the theory, and the other two [the identification (I.1.11) and cone (I.2.7) axioms] eliminate pathologies or excessive generality. In the present paper the axioms are supplemented by the nontriviality conditions (I.1.9) and a regularity property (II.4.2).

The starting point is a nonempty set M and a family of distinguished subsets, called *light rays*, which are totally ordered. The order axiom provides the properties of this order. Positive and negative cones at a point are defined in terms of increasing and decreasing subsets and are used to extend the total order on the light rays to a partial order over all of M. The first significant result is the *polygon lemma* (I.2.3) which provides an essential constructive tool. A non-topological definition is found for the interiors of the cones; it leads to a "more homogeneous" partial order relation on M.

In Sect. II, subsets called *D-sets* (Def. II.2.2), possessing certain desirable properties, are studied. The key concept of *perpendicularity* of light rays is isolated (Def. II.3.1) and used to derive the basic "separation properties," provided that the interiors of cones are nonempty. It is shown that, in a D-set, "good" properties of one cone can be transported along light rays, so that the structure of a D-set is homogeneous. In particular, if one cone has nonempty interior, so have all others. However, the existence of even one cone with nonempty interior does not follow from the axioms, but has to be imposed as an additional *regularity* condition. The local structure axiom now states that every point lies in a regular D-set. It is proved that the family of regular D-sets is closed under finite intersections. The *order topology* is defined as the topology which has this family as a base. This topology is Hausdorff, and coincides with the usual topology for Minkowski spaces.

0. Introduction

Ordered spaces play a central role in theoretical physics. The ordering of time, for instance, induces an ordering of the space of events by the future light cone.