Commun. Math. Phys. 132, 581–592 (1990)

Communications in Mathematical Physics

Implementation of Comparative Probability by Normal States. Infinite Dimensional Case

Simba A. Mutangadura

Department of Physics, University of Zimbabwe, Box M.P. 167, Mt. Pleasant, Harare, Zimbabwe* and International Centre for Theoretical Physics, Trieste, Italy

Abstract. Let $\mathscr{H}$ be an infinite dimensional Hilbert space and $\mathscr{P}(\mathscr{H})$ the set of all (orthogonal) projections on $\mathscr{H}$. A comparative probability on $\mathscr{P}(\mathscr{H})$ is a linear preorder $\lesssim$ on $\mathscr{P}(\mathscr{H})$ such that $\mathbf{O}\lesssim P\lesssim\mathbf{1}$, $\mathbf{1}\not\lesssim\mathbf{O}$ and such that if $P\perp R$, $Q\perp R$, then $P\lesssim Q\Leftrightarrow P+R\lesssim Q+R$ for all P, Q, R in $\mathscr{P}(\mathscr{H})$. We give a sufficient and necessary condition for $\lesssim$ to be implemented in a canonical way by a normal state on $\mathscr{B}(\mathscr{H})$, the bounded linear operators on $\mathscr{H}$.

1. Introduction and Notation

Let $\mathscr{H}$ be a Hilbert space. $\mathscr{P}(\mathscr{H})$ denotes the set of all (orthogonal) projections on $\mathscr{H}$. If E is a closed subspace of $\mathscr{H}$, and $\phi\in\mathscr{H}$ then P_E and P_ϕ denote the corresponding projections. We drop the E and ϕ if no reference to the subspaces is required. $\mathscr{P}_1(\mathscr{H})$ is the subset of all one dimensional projections and $\mathscr{P}_\sigma(\mathscr{H})$ is the subset of all those projections P_E such that E is a separable (finite or infinite dimensional) subspace of $\mathscr{H}$. Lower case Roman subscripts as in P_j or P_{ϕ_k} will generally be used for indexing sequences and nets. **N**, **R** and **C** denote the natural numbers, the reals and the complex numbers respectively. All vectors of $\mathscr{H}$ appearing may be assumed to be normalised. $P_{\mathscr{H}}$ is denoted by $\mathbf{1}_{\mathscr{H}}$ or just $\mathbf{1}$ of no confusion arises and the zero vector is denoted by $\mathbf{O}$. The orthogonal complement of P (i.e. $\mathbf{1}-P$) is denoted by $P^\perp$. If $P, Q\in\mathscr{P}(\mathscr{H})$ and $P\leqq Q^\perp$ then we write $P\perp Q$.

Definition 1.1. Let $\mathscr{H}$ be a Hilbert space. A preorder relation $\lesssim$ on $\mathscr{P}(\mathscr{H})$ is called an elementary comparative probability (ECP) iff the following axioms are satisfied by all P, Q, $R\in\mathscr{P}(\mathscr{H})$:

A1 $P\lesssim Q$ or $Q\lesssim P$,

A2 $P\lesssim Q$ and $Q\lesssim R\Rightarrow P\lesssim R$,

A3 $\mathbf{O}\lesssim P\lesssim\mathbf{1}$, $\mathbf{1}\not\lesssim\mathbf{O}$.

* Permanent address