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Multiple Forced Oscillations for the N-Pendulum Equation

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Abstract. We consider the periodically forced N-pendulum equation. Forced oscillations are obtained, and their multiplicity is studied in terms of the mean value of the forcing term.

Introduction

Let m_i, l_i be positive constants, $i=1,\ldots,N$, and set $M_j=\sum_{i=j}^{N} m_i$, $j=1,\ldots,N$. If $\theta=(\theta_1,\ldots,\theta_N)$, $\xi=(\xi_1,\ldots,\xi_N)\in\mathbb{R}^N$, then the Lagrangian

$$\mathscr{L}(\theta,\xi,t)=\frac{1}{2}\sum_{i,j=1}^{N} M_{\max(i,j)}\, l_i l_j \cos(\theta_i-\theta_j)\xi_i\xi_i + g\sum_{j=1}^{N} M_j l_j\cos\theta_j+\sum_{j=1}^{N} f_j(t)\theta_j$$

($g=$ constant of gravitation)

corresponds to the mechanical system of N coplanar penduli with masses m_k, length l_k subject to the forcing terms $f_k=f_k(t)$, $k=1,\ldots,N$. (Here θ_k is the angle of the k-pendulum with the vertical.) As is well known, the corresponding equations of motion are:

$$\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\xi_j}(\theta,\dot\theta,t)-\frac{\partial\mathscr{L}}{\partial\theta_j}(\theta,\dot\theta,t)=0,\quad j=1,\ldots,N. \tag{0.1}$$

Assuming the forcing terms are T-periodic (i.e. $f_k(t+T)=f_k(t)\ \forall t\in\mathbb{R}\ \forall k=1,\ldots,N$) we are interested in finding T-periodic solutions for (0.1).

Notice that this problem admits a natural $\mathbb{Z}^N$ symmetry, in the sense that if $\theta=\theta(t)$ is a T-periodic solution for (0.1) so is $\theta(t)+2\pi k\ \forall k\in\mathbb{Z}^N$. Therefore we shall call *distinct* the solutions of (0.1) whose difference does not belong to $2\pi\mathbb{Z}^N:=\{2\pi k\ \forall k\in\mathbb{Z}^N\}$. As pointed out by many authors (e.g. [5, 3, 1]), if the forcing terms f_k have *zero* mean value (i.e. $\int_0^T f_k=0$, $k=1,\ldots,N$) then this symmetry is preserved by the variational principle associated to (0.1). Namely, T-periodic solutions of

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