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The Heat Semigroup and Integrability of Lie Algebras: Lipschitz Spaces and Smoothness Properties

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Dedicated to Res Jost and Arthur Wightman

Abstract. We define and analyze Lipschitz spaces $\mathscr{B}_{\alpha,q}$ associated with a representation $x \in g \to V(x)$ of the Lie algebra g by closed operators $V(x)$ on the Banach space $\mathscr{B}$ together with a heat semigroup S. If the action of S satisfies certain minimal smoothness hypotheses with respect to the differential structure of $(\mathscr{B}, g, V)$ then the Lipschitz spaces support representations of g for which products $V(x)V(y)$ are relatively bounded by the Laplacian generating S. These regularity properties of the $\mathscr{B}_{\alpha,q}$ can then be exploited to obtain improved smoothness properties of S on $\mathscr{B}$. In particular C_4-estimates on the action of S automatically imply C_∞-estimates. Finally we use these results to discuss integrability criteria for $(\mathscr{B}, g, V)$.

1. Introduction

Let $(\mathscr{B}, g, V)$ be a representation of the Lie algebra g by a family of closable operators $V = \{V(x); x \in g\}$ acting on a dense invariant subspace $\mathscr{B}_\infty$ of the Banach space $\mathscr{B}$ and let

$$\Delta = -\sum_{i=1}^{d} V(x_i)^2$$

denote the Laplacian associated with the basis $x_1, \ldots, x_d$ of g. If the $V(x)$ satisfy the usual dissipation properties required for generators of continuous one-parameter groups then it follows from [BGJR] that $(\mathscr{B}_\infty, g, V)$ integrates to a continuous representation U of the corresponding connected Lie group G if, and only if, Δ is closable and its closure $\bar{\Delta}$ generates a continuous semigroup S satisfying certain smoothness properties. These latter properties are of two kinds; *range conditions* $S_t\mathscr{B} \subseteq \mathscr{B}_n$, where $\mathscr{B}_n$ is the common domain of all nth order monomials in the $V(x)_j$, and *boundedness conditions*

$$\|V(x_{i_1}) \ldots V(x_{i_n}) S_t\| = O(t^{-n/2})$$