Commun. Math. Phys. 132, 201–215 (1990)

Communications in
Mathematical
Physics

Symmetry Groups and Non-Abelian Cohomology

R. F. Streater*

Center for Transport Theory and Mathematical Physics, Department of Mathematics and Physics, Virginia Polytechnic Institute, Blacksburg, VA 24061, USA

Dedicated to Res Jost and Arthur Wightman

Abstract. We consider the implementation of symmetry groups of automorphisms of an algebra of observables in a reducible representation whose multipliers in general are non-commuting operators in the commutant of the representation. The multipliers obey a non-abelian cocycle relation which generalizes the 2-cohomology of the group. Examples are given from the theory of spin algebras and continuous tensor products. For type *I* representations we show that the multiplier can be chosen to lie in the centre, giving an isomorphism with abelian theory.

1. Introduction

We start with Wigner's formulation of symmetry in quantum mechanics [1], which was used with serene power for the Poincaré group $\mathbb{P}_+^\uparrow$ [2]. The states $\{\boldsymbol{\psi}\}$ of a system are taken to form the unit sphere $\mathscr{H}_1$ in a projective Hilbert space $\mathscr{H}$: so if $\mathscr{H}$ is a Hilbert space and $\psi \in \mathscr{H}$ with $\|\psi\| = 1$, then the state $\boldsymbol{\psi}$ is the unit ray through ψ:

$$\boldsymbol{\psi} = \{\lambda\psi : |\lambda| = 1, \lambda \in \mathbb{C}\} \in \mathscr{H}_1 . \tag{1.1}$$

We furnish $\mathscr{H}_1$ with a transition probability:

$$\mathscr{P}(\boldsymbol{\psi}, \boldsymbol{\phi}) = |\langle \psi, \phi \rangle|^2 , \quad \psi \in \boldsymbol{\psi}, \ \phi \in \boldsymbol{\phi} . \tag{1.2}$$

A symmetry operation is an isometry $\mathbf{U}$, that is, a bijection $\mathbf{U} : \mathscr{H}_1 \to \mathscr{H}_1$ preserving $\mathscr{P}$:

$$\mathscr{P}(\mathbf{U}\boldsymbol{\psi}, \mathbf{U}\boldsymbol{\gamma}) = \mathscr{P}(\boldsymbol{\psi}, \boldsymbol{\phi}) \quad \text{for all} \quad \boldsymbol{\psi}, \boldsymbol{\phi} \in \mathscr{H}_1 . \tag{1.3}$$

The set of isometries is a group, denoted $\mathrm{Aut}\,\mathscr{H}_1$.

* Permanent address: Department of Mathematics, King's College, Strand, London WC2R2LS, UK