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Ward Identities for Non-Commutative Geometry*

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Dedicated to Res Jost and Arthur Wightman

Abstract. We interpret the cocycle condition in quantum field theory as a set of integrated Ward identities for non-commutative geometry.

I. Basic Notions

The Wightman functions of a super-symmetric quantum field theory given by a super-trace functional have a geometric or cohomological interpretation. This was shown in joint work of the authors with Lesniewski [JLO1]. This property is summarized by the construction of a cocycle τ for the ∂-complex of entire cyclic cohomology, namely a solution to the equation

$$\partial\tau = 0. \tag{I.1}$$

Here τ is a time average of certain Euclidean Wightman functionals, and ∂ is a standard coboundary operator of non-commutative geometry, see [C, JLO1]. Furthermore, this natural cohomological interpretation can be generalized to the case when the Wightman functionals τ are constructed from a finite-temperature functional satisfying a super-version of the KMS condition of statistical mechanics [K, JLO2, JLWis]. In this case the super-trace (associated with finite volume theories) need not exist – only the super-KMS functional has to be defined.

In this note we investigate how Eq. (I.1) has a set of identities as its consequence, which can be interpreted as a symmetry of the Wightman functions. We call these identities "Ward identities for non-commutative geometry." Let us consider an example. Let γ denote the Z_2 grading in our theory, equal to $(-I)^{N_f}$ in models, and let $A \to A^\gamma = \gamma A \gamma$ denote the action of the grading on field operators. Let $A(t) = e^{-tH} A e^{tH}$ denote the propagation of A to Euclidean time t, and let $\langle A \rangle$ denote the expectation

$$\langle A \rangle = \mathrm{Str}(A e^{-H}), \tag{I.2}$$

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